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Massless and massive one-loop three-point functions in negative dimensional approach

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Abstract. In this article we present the complete massless and massive one-loop triangle diagram results using the negative dimensional integration method (NDIM). We consider the following cases: massless internal fields; one massive, two massive with the same mass m and three equal masses for the virtual particles. Our results are given in terms of hypergeometric and hypergeometric-type functions of the external momenta (and masses for the massive cases) where the propagators in the Feynman integrals are raised to arbitrary exponents and the dimension of the space-time is D. Our approach reproduces the known results; it produces other solutions as yet unknown in the literature as well. These new solutions occur naturally in the context of NDIM revealing a promising technique to solve Feynman integrals in quantum field theories.

1 Introduction

The study of scattering amplitudes in quantum electrodynamics (QED), quantum chromodynamics (QCD), and the standard model (SM) for electroweak interactions, as well as the renormalization group, asymptotic freedom, and other properties of perturbative quantum field theories, have required each time the computation of complex Feynman integrals. Therefore, the development of refined mathematical methods and approaches to deal with them have been studied and applied to various cases with varying success. Some of these are: dimensional regularization [1,2], use of the Mellin–Barnes representation for hypergeometric functions [3], the method of Gegenbauer polynomials [4], integration by parts [5] and several others [6–14].

Another integration method that has been developed in recent years and applied with great success is the negative dimensional integration method (NDIM) [15]. This method uses the analytic continuation of dimension D into negative values. One advantage for NDIM is that the complexities of performing D-dimensional integrals are transferred to a resolution of systems of linear algebraic equations. The NDIM approach was proven to be satisfactory when applied to the calculation of various types of Feynman diagrams at one- and two-loop levels as well as in non-covariant gauges such as the light-cone and Coulomb ones [16–19].

In this paper we use NDIM to obtain the complete set of solutions to one-loop triangle diagram, that occur in some processes such as interactions between Z particles, gluons, etc.; diagrams which become more and more significant in the precision measurements leading to the checking of the electroweak standard model, and search for the Higgs intermediate boson for example. Some cases from this vertex graph type were investigated with other approaches [3,20–26]. On the other hand, with the NDIM technique we are able to obtain all the possible different solutions to the one-loop triangle diagram in different kinematical regions of interest according to the convergence region considered. These solutions are expressed in terms of hypergeometric-type functions.

This work is organized as follows. In Sect. 1 we illustrate the use of NDIM to solve the one-loop massless triangle and point out its solutions in Appendix A. Massive cases with two null masses, one null and two equal masses and three equal masses are treated in Sect. 2 and the solutions presented in Appendices B, C and D respectively. Wherever possible, results are then compared to known ones in the pertinent literature (either in a specific kinematical region or for particular cases with specific values for the exponents of the propagators set to minus one). Finally in Sect. 3 we discuss the main results of the paper and present our concluding remarks.

2 One-loop massless triangle

In this section, we present the result for the one-loop massless triangle diagram (see Fig. 1) with two independent ex-

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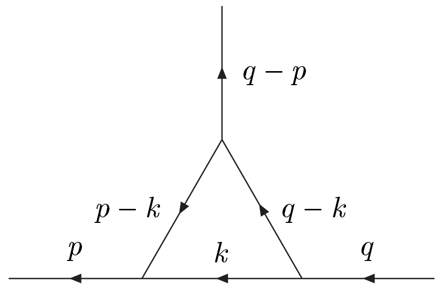

Fig. 1. One-loop massless triangle diagram

ternal momenta, evaluated according to the procedure of NDIM.

The most general form for the one-loop Feynman integral associated to this diagram is given by

$$\int \frac{\mathrm{d}^D k}{[k^2-m_1^2]^{a'}[(k-p)^2-m_2^2]^{b'}[(k-q)^2-m_3^2]^{c'}}, \quad (1)$$

where $D > 0$ and $a', b', c' > 0$.

In the context of NDIM we take the corresponding integral, namely,

$$\begin{aligned} J &= J(a,b,c,p,q,m_1,m_2,m_3) \quad (2) \\ &= \int \mathrm{d}^D k [k^2-m_1^2]^a [(k-p)^2-m_2^2]^b [(k-q)^2-m_3^2]^c, \end{aligned}$$

with $a, b, c > 0$, and $D < 0$. The physically interesting result emerges after analytic continuation into positive dimensionality and negative exponents for a, b, c.

We begin by taking the special case where all the internal field masses are set to zero; that is, we take $m_1 = m_2 = m_3 = 0$. The starting point is the evaluation of the corresponding Gaussian integral

$$\begin{aligned} I &= I(\alpha,\beta,\gamma,p,q) \\ &= \int \mathrm{d}^D k \exp\{-\alpha k^2 - \beta(k-p)^2 - \gamma(k-q)^2\} \quad (3) \\ &= \left[\frac{\pi}{\alpha+\beta+\gamma}\right]^{D/2} \\ &\quad \times \exp\left\{\frac{-\beta\gamma(p-q)^2 - \alpha\beta p^2 - \alpha\gamma q^2}{\alpha+\beta+\gamma}\right\}, \quad (4) \end{aligned}$$

where, after the expansion in α, β and γ powers, we get

$$\begin{aligned} I &= \pi^{D/2} \sum_{j_1,\cdots,j_6=0}^{\infty} (-1)^{j_1+j_2+j_3} \Gamma(1-j_1-j_2-j_3-D/2) \\ &\quad \times \frac{\alpha^{j_1+j_2+j_4}}{j_4!} \frac{\beta^{j_1+j_3+j_5}}{j_5!} \frac{\gamma^{j_2+j_3+j_6}}{j_6!} \frac{(p^2)^{j_1}}{j_1!} \frac{(q^2)^{j_2}}{j_2!} \frac{(r^2)^{j_3}}{j_3!}. \quad (5) \end{aligned}$$

Since we use a multinomial expansion the sum indices above are constrained by

$$D/2 = -\sum_{n=1}^{n=6} j_n.$$

On the other hand, expanding the exponential of (3), we have

$$\begin{aligned} I &= \sum_{a,b,c=0}^{\infty} (-1)^{a+b+c} \frac{\alpha^a \beta^b \gamma^c}{a!b!c!} \\ &\quad \times J(a,b,c,p,q,m_1=m_2=m_3=0), \quad (6) \end{aligned}$$

where, comparing the expressions (5) and (6) by its α, β and γ powers, we obtain a general relation for the integral $J = J(a,b,c,p,q)$:

$$\begin{aligned} J &= \pi^{D/2}(-1)^{-a-b-c}\Gamma(1+a)\Gamma(1+b)\Gamma(1+c) \\ &\quad \times \sum_{j_1,\cdots,j_6=0}^{\infty} (-1)^{j_1+j_2+j_3} \frac{\Gamma(1-j_1-j_2-j_3-D/2)}{j_4!j_5!j_6!} \\ &\quad \times \frac{(p^2)^{j_1}}{j_1!} \frac{(q^2)^{j_2}}{j_2!} \frac{(r^2)^{j_3}}{j_3!}, \quad (7) \end{aligned}$$

where for convenience we introduced $r = q - p$. From the comparison of powers we obtain three additional constraint equations besides the one already defined above originating from the multinomial expansion. Then, all constraints are

$$a = j_1 + j_2 + j_4, \quad (8)$$
$$b = j_1 + j_3 + j_5, \quad (9)$$
$$c = j_2 + j_3 + j_6, \quad (10)$$
$$\frac{D}{2} = -j_1 - j_2 - j_3 - j_4 - j_5 - j_6, \quad (11)$$

Therefore, there are four constraint equations with six variables. These form a system of linear equations that can only be solved if we leave two free indices and the result will be given as a double series. There are $C_4^6 = 15$ distinct ways in which we can choose these two free indices. From these 15 ways there are 12 with non-trivial solutions, since for three of the resulting systems of linear equations, the determinant is zero. These 12 solutions are grouped into three sets, each one with four solutions according to the kinematical configuration of the variables defined by the external momenta. Performing the analytic continuation to $D > 0$ and $a, b, c < 0$ we get the three sets of solutions for the relevant Feynman integral (see Appendix A, massless cases) according to their variables, where J_n, $n = 1, 2, \cdots, 12$, are written in terms of the Appel hypergeometric functions F_4. The first set of solutions above is in accordance with the result calculated in [3]. The other two new solutions represent other kinematical configuration regions not explored explicitly in text books but with the same physical importance. These two other hypergeometric series representations for the scalar integral in question can easily be obtained from set 1 by interchanging two external legs and the associated exponents of propagators, e.g., $p \leftrightarrow q$ and $b \leftrightarrow c$. However, the negative dimensional approach can generate a hypergeometric series which *cannot* be obtained through the symmetry of the diagram. An illustration of this occurs in the scalar one-loop box integral [16] for photon–photon scattering, where two of such solutions (which are not related

Table 1.

Set	Solutions
1	$J_1 + J_2 + J_3 + J_4$
2	$J_5 + J_6 + J_7 + J_8$
3	$J_9 + J_{10} + J_{11} + J_{12}$

by symmetry but through direct analytic continuation) are

$$F_3(\cdots|x, y) \quad \text{and} \quad H_2(\cdots.|x, -y^{-1}).$$

where F_3 and H_2 are the usual hypergeometric functions of two variables.

In the next cases, some hypergeometric series can also be obtained by symmetry considerations when one of the solutions is known (of course, no such symmetric solutions can ever be generated if one has nothing to start with!). Upon this given solution we can carry out the needed interchanges in momenta and exponents of the propagators in order to arrive at the other solutions. We emphasize, however, that besides these solutions which are connected by symmetry, there are completely new solutions which cannot be related by such means and these were not obtained previously by any other method in spite of the fact that they have the same physical importance. This is the strength of the NDIM technique where we obtain *simultaneously* all the solutions: those which are related simply by symmetry of the diagram and also those which are related by analytic continuation.

For the sake of completeness let us explicitly write one of these three hypergeometric series representations for the massless triangle integral. From Table 1 we choose set number 3, i.e.,

$$J(a,b,c,p,q) = A_9 F_4^{(9)} + A_{10} F_4^{(10)} + A_{11} F_4^{(11)} + A_{12} F_4^{(12)}, \qquad (12)$$

where the factors A_n are products of gamma functions and the $F_4^{(n)}$ are Appel hypergeometric functions whose parameters and variables are given in Table 5 in Appendix A, so that we have

$$\begin{aligned} J(a,b,c,p,q) &= \pi^{D/2}(p^2)^\sigma \\ &\times \Bigg\{ (z_1)^{\sigma-a}(z_2)^{\sigma-b} \frac{(-a)_{-c-D/2}(-b)_{-c-D/2}}{(c+D/2)_{-2c-D/2}} \\ &\times F_4 \left[\begin{matrix} x_1^{(9)}, x_2^{(9)} \\ x_3^{(9)}, x_4^{(9)} \end{matrix} \middle| z_1; z_2 \right] \\ &+ (z_2)^{\sigma-b} \frac{(-a)_\sigma(-c)_{-b+\sigma}}{(b-\sigma)_{-b+2\sigma+D/2}} F_4 \left[\begin{matrix} x_1^{(10)}, x_2^{(10)} \\ x_3^{(10)}, x_4^{(10)} \end{matrix} \middle| z_1; z_2 \right] \\ &+ (z_1)^{\sigma-a} \frac{(-b)_\sigma(-c)_{-a+\sigma}}{(a-\sigma)_{-a+2\sigma+D/2}} F_4 \left[\begin{matrix} x_1^{(11)}, x_2^{(11)} \\ x_3^{(11)}, x_4^{(11)} \end{matrix} \middle| z_1; z_2 \right] \\ &+ \frac{(-a)_\sigma(-b)_\sigma}{(-\sigma)_{2\sigma+D/2}} F_4 \left[\begin{matrix} x_1^{(12)}, x_2^{(12)} \\ x_3^{(12)}, x_4^{(12)} \end{matrix} \middle| z_1; z_2 \right] \Bigg\}, \end{aligned} \qquad (13)$$

Table 2.

Set	Solutions	Set	Solutions	Set	Solutions	Set	Solutions
1	$J_1 + J_2$	5	$J_7 + J_8$	9	J_{12}	13	J_{18}
2	J_3	6	J_9	10	$J_{13} + J_{14}$	14	$J_{19} + J_{20}$
3	J_4	7	J_{10}	11	$J_{15} + J_{16}$	15	$J_{21} + J_{22}$
4	$J_5 + J_6$	8	J_{11}	12	J_{17}	16	J_{23}

with $\sigma = a + b + c + D/2$ and in this particular set, $z_1 = r^2/p^2 = (q-p)^2/p^2$, $z_2 = q^2/p^2$ and the parameters $x_1^{(9)}, \cdots, x_4^{(12)}$ can be read off from Table 5 in Appendix A:

$$\begin{aligned} x_1^{(9)} &= c + D/2, \quad x_2^{(9)} = \sigma + D/2; \\ x_3^{(9)} &= 1 - a + \sigma, \quad x_4^{(9)} = 1 - b + \sigma; \\ x_1^{(10)} &= -b, \quad x_2^{(10)} = a + D/2; \\ x_3^{(10)} &= 1 + a - \sigma, \quad x_4^{(10)} = 1 - b + \sigma; \\ x_1^{(11)} &= -a, \quad x_2^{(11)} = b + D/2; \\ x_3^{(11)} &= 1 - a + \sigma, \quad x_4^{(11)} = 1 + b - \sigma; \\ x_1^{(12)} &= -c, \quad x_2^{(12)} = -\sigma; \\ x_3^{(12)} &= 1 + a - \sigma, \quad x_4^{(12)} = 1 + b - \sigma. \end{aligned} \qquad (14)$$

The symmetry of the original integral allows us to obtain two other hypergeometric series representations, just interchanging external legs and exponents of the propagators. These are written down in Appendix A. The first set, $A_1F_4^{(1)} + A_2F_4^{(2)} + A_3F_4^{(3)} + A_4F_4^{(4)}$, which is also a sum of four F_4 Appel series was obtained by Boos and Davydychev in [3], see (23) of their paper. The correspondence with our result is achieved by using $(a \leftrightarrow -\mu, b \leftrightarrow -\nu, c \leftrightarrow -\rho)$.

2.1 One-loop triangle, one mass case

Now we analyze a second case that occurs when there is only one massive propagator within the Feynman integral represented by (1) with $m_1 = m$ and $m_2 = m_3 = 0$. Again, we used the negative dimensional integral method to obtain twenty-three non-trivial solutions with sixteen sets of independent solutions for the integral, given in terms of hypergeometric functions of three variables, shown in Appendix B (one massive propagator case). The sixteen independent solutions are shown in Table 2.

The first solution, namely, $J_1 + J_2$, is in accordance with the result calculated in [3]. Note that there are fifteen other novel results defined in different kinematical regions not computed anywhere else.

In order to understand better the notations used in our paper, we will write one of these new series representing the Feynman triangle integral with one mass. From Appendix B, Tables 7 and 8, we observe that every hypergeometric series representation for the triangle with one massive propagator is of the form J = factor × series, where the poles (if they exist) are contained in the factor and the series is given by one of the $T_1, \cdots, T_8$.

Let us choose, for instance, one of the series that was not known in the literature, namely, J_4. From Table 2, the solution of set 3 gives exactly J_4 and we observe that using Tables 7 and 8, Appendix B,

$$\begin{aligned} &J_4(a,b,c,p,q,m,0,0) = B_4T_2 \\ &= \pi^{D/2}(r^2)^\sigma(z_1)^{\sigma-b}(z_2)^{\sigma-c}\frac{(-b)_{-a-D/2}(-c)_{-a-D/2}}{(a+D/2)_{-2a-D/2}} \\ &\times T_2\left[\begin{array}{c|c} x_1,x_2,x_3 & \\ x_4,x_5 & \end{array} z_1;z_2;z_3\right], \end{aligned} \tag{15}$$

where again $\sigma = a+b+c+D/2$, and the parameters are given in Table 8 ($n=4$), i.e.,

$$x_1 = a+D/2, \quad x_2 = \sigma+D/2, \quad x_3 = 1-\sigma-D/2;$$
$$x_4 = 1-b+\sigma, \quad x_5 = 1-c+\sigma,$$

and also the variables

$$z_1 = \frac{q^2}{r^2}, \quad z_2 = \frac{p^2}{r^2}, \quad z_3 = -\frac{m^2r^2}{p^2q^2}.$$

The series T_2 is defined in Appendix B.

2.2 One-loop triangle, two equal masses case

We consider also the case of two massive propagators with equal masses, where $m_1 = 0$ and $m_2 = m_3 = m$. We have obtained thirty-two independent solutions expressed by the hypergeometric functions of three and four variables (see Appendix C, two massive propagators). The solution J_{32} is in accordance with the result calculated in [3]. Note again that there are thirty-one other solutions in different kinematical regions.

Also, in order to understand better the notations used we write explicitly one of such new hypergeometric series representations, which is a 4-fold series. From Appendix C, Tables 10 and 11, we have

$$\begin{aligned} &J_1(a,b,c,p,q,0,m,m) \\ &= \pi^{D/2}(p^2)^\sigma\frac{(-a)_\sigma(-b)_\sigma}{(-\sigma)_{2\sigma+D/2}}R_1\left[\begin{array}{c|c} x_1,x_2,x_3 & \\ x_4,x_5 & \end{array} z_1;z_2;z_3;z_4\right], \end{aligned} \tag{16}$$

where the parameters and variables can be read off from Table 11, Appendix C, i.e.,

$$x_1 = -\sigma, \quad x_2 = -c, \quad x_3 = 1-\sigma-D/2;$$
$$x_4 = 1+a-\sigma, \quad x_5 = 1+b-\sigma,$$

and

$$z_1 = -\frac{m^2}{p^2}, \quad z_2 = \frac{q^2}{p^2}, \quad z_3 = \frac{r^2}{p^2}, \quad z_4 = \frac{m^2}{p^2}.$$

The series R_1 is defined in Appendix C.

Table 3.

Set	Solutions	Set	Solutions	Set	Solutions
1	J_1	13	J_{16}	25	$J_{28}+J_{29}$
2	J_2	14	J_{17}	26	$J_{30}+J_{31}$
3	J_3	15	J_{18}	27	$J_{32}+J_{33}$
4	J_4	16	J_{19}	28	J_{34}
5	J_5	17	J_{20}	29	J_{35}
6	J_6	18	J_{21}	30	J_{36}
7	J_7	19	J_{22}	31	J_{37}
8	J_8	20	J_{23}	32	J_{38}
9	J_9	21	J_{24}	33	J_{39}
10	$J_{10}+J_{11}$	22	J_{25}	34	J_{40}
11	$J_{12}+J_{13}$	23	J_{26}		
12	$J_{14}+J_{15}$	24	J_{27}		

2.3 One-loop triangle, three equal masses case

Finally, we consider the Feynman integral with three equal masses ($m_1 = m_2 = m_3 = m$) and obtain forty solutions, which are grouped into thirty-four independent solutions (see Table 3), also expressed through the hypergeometric functions of three, four and five variables (see Appendix D, three massive denominators).

The last solution, namely J_{40}, is in accordance with the result calculated in [3].

Let us write down the first one as an example. From Appendix D, Tables 13 and 14, we have

$$\begin{aligned} &J_1(a,b,c,p,q,\{m_i\}) \\ &= \pi^{D/2}(r^2)^\sigma(-z_4)^{\sigma-c}(-z_5)^{\sigma-b}\frac{(-b)_{-a-D/2}(-c)_{-a-D/2}}{(a+D/2)_{-2a-D/2}} \\ &\times \Psi_1\left[\begin{array}{c|c} x_1,x_2,x_3 & \\ x_4,x_5 & \end{array} z_1;z_2;z_3;z_4;z_5\right], \end{aligned} \tag{17}$$

where the parameters and variables are, respectively,

$$x_1 = 1-\sigma-D/2, \quad x_2 = c-\sigma, \quad x_3 = b-\sigma;$$
$$x_4 = 1-a-D/2, \quad x_5 = 1-\sigma-D/2,$$

and

$$z_1 = -\frac{m^2r^2}{p^2q^2}, \quad z_2 = \frac{m^2}{p^2}, \quad z_3 = \frac{m^2}{q^2},$$
$$z_4 = -\frac{p^2}{r^2}, \quad z_5 = -\frac{q^2}{r^2}.$$

The series Ψ_1 is defined in Appendix D.

3 Conclusion

We pointed out that our new results, written as multiple hypergeometric series, are related to each other through analytic continuation, direct or indirect. If that is the case, then we could ask: If we had one of these solutions, would it be possible to obtain all the others using analytic continuation formulas? The answer is, of course, affirmative and one would say *yes*, in principle. However, we also need to point out that very few of these formulas for analytic continuation are known for multiple hypergeometric series; most of the known formulas for the analytic continuation are restricted to the simplest of them, namely, ${}_2F_1(a,b;c|z)$, which is a single series. Some formulas for Appel's functions are also known [27], and even for Meijer's functions. However, for three and higher variables (or triple and n-fold) hypergeometric series we do not have known analytic continuation formulas relating $z_j \to z_j^{-1}$, or $z_j \to 1 - z_j$ etc. This is precisely the case in quantum field theory: multiple hypergeometric series that obey *unknown* analytic continuation formulas. We can go back to the question raised at the beginning of this section and correct the answer accordingly: the answer is indeed *yes* once and if one knows the formulas for such a continuation! So we propose also that NDIM is a technique where one can infer such formulas, because all of the results we obtain *simultaneously* represent the same integral, so all of them are related through analytic continuation, from the very definition of analytic functions and analytic continuation.

In this work we have used the NDIM approach to evaluate the one-loop triangle diagram with massless and massive (with one, two and three equal masses) internal particles. These kinds of diagrams are relevant to, e.g., vertex corrections in QED and QCD, Z-scattering in electroweak interactions and so on. For the massless case, we have three distinct kinematical regions obtained simultaneously; for the one mass case we have sixteen distinct kinematical regions; thirty-two for the two equal masses case and thirty-six for the three equal masses case. All solutions are represented by hypergeometric-type functions of several variables and some of these were compared to existing results in the literature and in all such cases our results do match the known ones. Therefore we deem NDIM as a powerful technique in the computation of highly complex Feynman integrals in various kinematical regions of interest.

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Appendix

A Massless case

The massless solutions $J_n = J_n(a,b,c,D,p,q)$, where $n = 1,2,\cdots,12$, are given by the Appel hypergeometric function

$$F_4\left[x_1,x_2;x_3,x_4|z_1;z_2\right] = \sum_{i,j=0}^{\infty} \frac{(x_1)_{i+j}(x_2)_{i+j}}{(x_3)_i(x_4)_j} \frac{z_1^i}{i!} \frac{z_2^j}{j!},$$

and the general expression for the solutions are given by $J_n = A_n F_4$, where the coefficients A_n are shown in Table 4 and the parameters and variables of the functions F_4 in Table 5. When appropriate, we use lines between two subsequent sets of solutions in the tables to separate different kinematical regions. Also, wherever convenient, we set $\sigma = a+b+c+D/2$.

Table 4.

n	A_n
1	$\pi^{D/2}(r^2)^\sigma(z_1)^{\sigma-b}(z_2)^{\sigma-c}\frac{(-b)_{-a-D/2}(-c)_{-a-D/2}}{(a+D/2)_{-2a-D/2}}$
2	$\pi^{D/2}(r^2)^\sigma(z_2)^{\sigma-c}\frac{(-a)_{-c+\sigma}(-b)_\sigma}{(c-\sigma)_{-c+2\sigma+D/2}}$
3	$\pi^{D/2}(r^2)^\sigma(z_1)^{\sigma-b}\frac{(-a)_{-b+\sigma}(-c)_\sigma}{(b-\sigma)_{-b+2\sigma+D/2}}$
4	$\pi^{D/2}(r^2)^\sigma\frac{(-b)_\sigma(-c)_\sigma}{(-\sigma)_{2\sigma+D/2}}$
5	$\pi^{D/2}(q^2)^\sigma(z_1)^{\sigma-a}(z_2)^{\sigma-c}\frac{(-a)_{-b-D/2}(-c)_{-b-D/2}}{(b+D/2)_{-2b-D/2}}$
6	$\pi^{D/2}(q^2)^\sigma(z_2)^{\sigma-c}\frac{(-a)_\sigma(-b)_{-c+\sigma}}{(c-\sigma)_{-c+2\sigma+D/2}}$
7	$\pi^{D/2}(q^2)^\sigma(z_1)^{\sigma-a}\frac{(-b)_{-a+\sigma}(-c)_\sigma}{(a-\sigma)_{-a+2\sigma+D/2}}$
8	$\pi^{D/2}(q^2)^\sigma\frac{(-a)_\sigma(-c)_\sigma}{(-\sigma)_{2\sigma+D/2}}$
9	$\pi^{D/2}(p^2)^\sigma(z_1)^{\sigma-a}(z_2)^{\sigma-b}\frac{(-a)_{-c-D/2}(-b)_{-c-D/2}}{(c+D/2)_{-2c-D/2}}$
10	$\pi^{D/2}(p^2)^\sigma(z_2)^{\sigma-b}\frac{(-a)_\sigma(-c)_{-b+\sigma}}{(b-\sigma)_{-b+2\sigma+D/2}}$
11	$\pi^{D/2}(p^2)^\sigma(z_1)^{\sigma-a}\frac{(-b)_\sigma(-c)_{-a+\sigma}}{(a-\sigma)_{-a+2\sigma+D/2}}$
12	$\pi^{D/2}(p^2)^\sigma\frac{(-a)_\sigma(-b)_\sigma}{(-\sigma)_{2\sigma+D/2}}$

Table 5.

n	$x_1, x_2; x_3, x_4$	z_1, z_2
1	$a+D/2,\ \sigma+D/2;\ 1-b+\sigma,\ 1-c+\sigma$	$\frac{q^2}{r^2}, \frac{p^2}{r^2}$
2	$-c,\ b+D/2;\ 1+b+\sigma,\ 1-c+\sigma$	$\frac{q^2}{r^2}, \frac{p^2}{r^2}$
3	$-b,\ c+D/2;\ 1-b+\sigma,\ 1+c-\sigma$	$\frac{q^2}{r^2}, \frac{p^2}{r^2}$
4	$-a,\ -\sigma;\ 1+b-\sigma,\ 1+c-\sigma$	$\frac{q^2}{r^2}, \frac{p^2}{r^2}$
5	$b+D/2,\ \sigma+D/2;\ 1-a+\sigma,\ 1-c+\sigma$	$\frac{r^2}{q^2}, \frac{p^2}{q^2}$
6	$-c,\ a+D/2;\ 1+a-\sigma,\ 1-c+\sigma$	$\frac{r^2}{q^2}, \frac{p^2}{q^2}$
7	$-a,\ c+D/2;\ 1-a+\sigma,\ 1+c-\sigma$	$\frac{r^2}{q^2}, \frac{p^2}{q^2}$
8	$-b,\ -\sigma;,1+a-\sigma,\ 1+c-\sigma$	$\frac{r^2}{q^2}, \frac{p^2}{q^2}$
9	$c+D/2,\ \sigma+D/2;\ 1-a+\sigma,\ 1-b+\sigma$	$\frac{r^2}{p^2}, \frac{q^2}{p^2}$
10	$-b,\ a+D/2;\ 1+a-\sigma,\ 1-b+\sigma$	$\frac{r^2}{p^2}, \frac{q^2}{p^2}$
11	$-a,\ b+D/2;\ 1-a+\sigma,\ 1+b-\sigma$	$\frac{r^2}{p^2}, \frac{q^2}{p^2}$
12	$-c,\ -\sigma;\ 1+a-\sigma,\ 1+b-\sigma$	$\frac{r^2}{p^2}, \frac{q^2}{p^2}$

B One massive propagator

The one massive denominator solutions $J_n = J_n(a,b,c,D,p,q,m,0,0)$, where $n = 1,2,\cdots,23$, are given by hy-

Table 6.

n	$1,2,3$	4	$5,\cdots,8$	$9,\cdots,12$	$13,\cdots,16$	$17,18$	$19,\cdots,22$	23
l	1	2	3	4	5	6	7	8

pergeometric functions listed below:

$$T_1\left[\left.\begin{matrix}x_1,x_2,x_3\\ x_4,x_5\end{matrix}\right| z_1;z_2;z_3\right] = \sum_{j_1,j_2,j_3=0}^{\infty} \frac{(x_1)_{j_1+j_2+j_3}(x_2)_{j_1+j_3}(x_3)_{j_2+j_3}}{(x_4)_{j_1+j_2+j_3}(x_5)_{j_3}} \frac{z_1^{j_1}}{j_1!}\frac{z_2^{j_2}}{j_2!}\frac{z_3^{j_3}}{j_3!},$$

$$T_2\left[\left.\begin{matrix}x_1,x_2,x_3\\ x_4,x_5\end{matrix}\right| z_1;z_2;z_3\right] = \sum_{j_1,j_2,j_3=0}^{\infty} \frac{(x_1)_{j_1+j_2-j_3}(x_2)_{j_1+j_2-j_3}(x_3)_{j_3}}{(x_4)_{j_1-j_3}(x_5)_{j_2-j_3}} \frac{z_1^{j_1}}{j_1!}\frac{z_2^{j_2}}{j_2!}\frac{z_3^{j_3}}{j_3!},$$

$$T_3\left[\left.\begin{matrix}x_1,x_2,x_3\\ x_4,x_5\end{matrix}\right| z_1;z_2;z_3\right] = \sum_{j_1,j_2,j_3=0}^{\infty} \frac{(x_1)_{j_1+j_2-j_3}(x_2)_{j_1+j_2}(x_3)_{j_3}}{(x_4)_{j_2-j_3}(x_5)_{j_1}} \frac{z_1^{j_1}}{j_1!}\frac{z_2^{j_2}}{j_2!}\frac{z_3^{j_3}}{j_3!},$$

$$T_4\left[\left.\begin{matrix}x_1,x_2,x_3\\ x_4,x_5\end{matrix}\right| z_1;z_2;z_3\right] = \sum_{j_1,j_2,j_3=0}^{\infty} \frac{(x_1)_{j_1+j_2}(x_2)_{j_1+j_2}(x_3)_{j_1-j_3}}{(x_4)_{j_1-j_3}(x_5)_{j_1+j_2-j_3}} \frac{z_1^{j_1}}{j_1!}\frac{z_2^{j_2}}{j_2!}\frac{z_3^{j_3}}{j_3!},$$

$$T_5\left[\left.\begin{matrix}x_1,x_2,x_3\\ x_4,x_5\end{matrix}\right| z_1;z_2;z_3\right] = \sum_{j_1,j_2,j_3=0}^{\infty} \frac{(x_1)_{j_1+j_2}(x_2)_{j_1+j_3}(x_3)_{j_2-j_3}}{(x_4)_{j_2-j_3}(x_5)_{j_1}} \frac{z_1^{j_1}}{j_1!}\frac{z_2^{j_2}}{j_2!}\frac{z_3^{j_3}}{j_3!},$$

$$T_6\left[\left.\begin{matrix}x_1,x_2,x_3\\ x_4,x_5\end{matrix}\right| z_1;z_2;z_3\right] = \sum_{j_1,j_2,j_3=0}^{\infty} \frac{(x_1)_{j_1+j_2}(x_2)_{j_1+j_2}(x_3)_{j_3}}{(x_4)_{j_2-j_3}(x_5)_{j_1+j_3}} \frac{z_1^{j_1}}{j_1!}\frac{z_2^{j_2}}{j_2!}\frac{z_3^{j_3}}{j_3!},$$

$$T_7\left[\left.\begin{matrix}x_1,x_2,x_3\\ x_4,x_5\end{matrix}\right| z_1;z_2;z_3\right] = \sum_{j_1,j_2,j_3=0}^{\infty} \frac{(x_1)_{j_1+j_2+j_3}(x_2)_{j_1+j_2}(x_3)_{j_3}}{(x_4)_{j_1+j_3}(x_5)_{j_2}} \frac{z_1^{j_1}}{j_1!}\frac{z_2^{j_2}}{j_2!}\frac{z_3^{j_3}}{j_3!},$$

$$T_8\left[\left.\begin{matrix}x_1,x_2,x_3\\ x_4,x_5\end{matrix}\right| z_1;z_2;z_3\right] = \sum_{j_1,j_2,j_3=0}^{\infty} \frac{(x_1)_{j_1+j_2+j_3}(x_2)_{j_1+j_2+j_3}(x_3)_{j_3}}{(x_4)_{j_2+j_3}(x_5)_{j_1+j_3}} \frac{z_1^{j_1}}{j_1!}\frac{z_2^{j_2}}{j_2!}\frac{z_3^{j_3}}{j_3!},$$

and the expression of each one solution is given by $J_n = B_nT_l$, where the relation between n and l is given by Table 6, where the coefficients B_n are shown in Table 7, the

Table 7.

n	B_n
1	$\pi^{D/2}(-m^2)^\sigma(z_3)^{\sigma-a}\frac{(-b)_{-a+\sigma}(-c)_{-a+\sigma}}{(a-\sigma)_{-2a+2\sigma+D/2}}$
2	$\pi^{D/2}(-m^2)^\sigma\frac{(-a)_\sigma}{(-\sigma)_{\sigma+D/2}}$
3	$\pi^{D/2}(-m^2)^\sigma(-z_1)^{-b}(-z_2)^{-c}\frac{(-a)_\sigma}{(-a-D/2)_{\sigma+D/2}}$
4	$\pi^{D/2}(r^2)^\sigma(z_1)^{\sigma-b}(z_2)^{\sigma-c}\frac{(-b)_{-a-D/2}(-c)_{-a-D/2}}{(a+D/2)_{-2a-D/2}}$
5	$\pi^{D/2}(q^2)^\sigma(z_1)^{\sigma-a}(z_2)^{\sigma-c}\frac{(-a)_{-b-D/2}(-c)_{-b-D/2}}{(b+D/2)_{-2b-D/2}}$
6	$\pi^{D/2}(q^2)^\sigma(z_2)^{\sigma-c}\frac{(-a)_\sigma(-b)_{-c+\sigma}}{(c-\sigma)_{-c+2\sigma+D/2}}$
7	$\pi^{D/2}(p^2)^\sigma(z_1)^{\sigma-a}(z_2)^{\sigma-b}\frac{(-a)_{-c-D/2}(-b)_{-c-D/2}}{(c+D/2)_{-2c-D/2}}$
8	$\pi^{D/2}(p^2)^\sigma(z_2)^{\sigma-b}\frac{(-a)_\sigma(-c)_{-b+\sigma}}{(b-\sigma)_{-b+2\sigma+D/2}}$
9	$\pi^{D/2}(p^2)^\sigma(z_2)^{-c}(z_3)^{\sigma-b}\frac{(-a)_{-c+\sigma}(-b)_c}{(b-\sigma)_{\sigma+D/2}}$
10	$\pi^{D/2}(q^2)^\sigma(z_2)^{-b}(z_3)^{\sigma-c}\frac{(-a)_{-b+\sigma}(-c)_b}{(c-\sigma)_{\sigma+D/2}}$
11	$\pi^{D/2}(r^2)^\sigma(z_1)^{\sigma-c}\frac{(-a)_{-c+\sigma}(-b)_c}{(c-\sigma)_{\sigma+D/2}}$
12	$\pi^{D/2}(r^2)^\sigma(z_1)^{\sigma-b}\frac{(-a)_{-b+\sigma}(-c)_b}{(b-\sigma)_{\sigma+D/2}}$
13	$\pi^{D/2}(p^2)^\sigma(z_1)^{\sigma-a}(-z_2)^{\sigma-b}\frac{(-a)_{-c-D/2}(-b)_{-c-D/2}}{(c+D/2)_{-2c-D/2}}$
14	$\pi^{D/2}(p^2)^\sigma(-z_2)^{\sigma-b}\frac{(-a)_\sigma}{(b-\sigma)_{\sigma+D/2}}$
15	$\pi^{D/2}(q^2)^\sigma(z_1)^{\sigma-a}(-z_2)^{\sigma-c}\frac{(-a)_{-b-D/2}(-c)_{-b-D/2}}{(b+D/2)_{-2b-D/2}}$
16	$\pi^{D/2}(q^2)^\sigma(-z_2)^{\sigma-c}\frac{(-a)_\sigma}{(c-\sigma)_{\sigma+D/2}}$
17	$\pi^{D/2}(r^2)^\sigma(z_2)^{\sigma-c}\frac{(-a)_{-c+\sigma}(-b)_\sigma}{(c-\sigma)_{-c+2\sigma+D/2}}$
18	$\pi^{D/2}(r^2)^\sigma(z_2)^{\sigma-b}\frac{(-a)_{-b+\sigma}(-c)_\sigma}{(b-\sigma)_{-b+2\sigma+D/2}}$
19	$\pi^{D/2}(p^2)^\sigma(z_2)^{\sigma-a}\frac{(-b)_\sigma(-c)_{-a+\sigma}}{(a-\sigma)_{-a+2\sigma+D/2}}$
20	$\pi^{D/2}(p^2)^\sigma\frac{(-a)_\sigma(-b)_\sigma}{(-\sigma)_{2\sigma+D/2}}$
21	$\pi^{D/2}(q^2)^\sigma(z_2)^{\sigma-a}\frac{(-b)_{-a+\sigma}(-c)_\sigma}{(a-\sigma)_{-a+2\sigma+D/2}}$
22	$\pi^{D/2}(q^2)^\sigma\frac{(-a)_\sigma(-c)_\sigma}{(-\sigma)_{2\sigma+D/2}}$
23	$\pi^{D/2}(r^2)^\sigma\frac{(-b)_\sigma(-c)_\sigma}{(-\sigma)_{2\sigma+D/2}}$

parameters and variables of the functions T_l in Table 8 (consider $\sigma = a+b+c+D/2$).

C Two massive propagators

The two massive denominators solutions $J_n = J_n(a,b,c,D,p,q,0,m,m)$, where $n = 1,2,\cdots,32$, are given by hypergeometric functions listed below:

$$R_1\left[\left.\begin{matrix}x_1,x_2,x_3\\ x_4,x_5\end{matrix}\right| z_1;z_2;z_3;z_4\right]$$

Table 8.

n	$x_1, x_2, x_3;\ x_4, x_5$	z_1, z_2, z_3
1	$-a,\ c+D/2,\ b+D/2; \sigma-a+D/2,\ 1-a+\sigma$	$\frac{p^2}{m^2};\ \frac{q^2}{m^2};\ -\frac{r^2}{m^2}$
2	$-\sigma,\ -b,\ -c; D/2,\ 1+a-\sigma$	$\frac{p^2}{m^2};\ \frac{q^2}{m^2};\ -\frac{r^2}{m^2}$
3	$1+a-\sigma,\ -b,\ -c;\ 1+a+D/2,\ 1+a-\sigma$	$\frac{m^2}{p^2};\ \frac{m^2}{q^2};\ -\frac{m^2r^2}{p^2q^2}$
4	$a+D/2,\ \sigma+D/2,\ 1-\sigma-D/2;\ 1-b+\sigma,\ 1-c+\sigma$	$\frac{q^2}{r^2};\ \frac{p^2}{r^2};\ -\frac{m^2r^2}{p^2q^2}$
5	$\sigma+D/2,\ b+D/2,\ 1-\sigma-D/2; 1-c+\sigma,\ 1-a+\sigma$	$\frac{r^2}{q^2};\ \frac{p^2}{q^2};\ \frac{m^2}{p^2}$
6	$a+D/2,\ -c,\ 1-\sigma-D/2;\ 1-c+\sigma,\ 1+a-\sigma$	$\frac{r^2}{q^2};\ \frac{p^2}{q^2};\ \frac{m^2}{p^2}$
7	$\sigma+D/2,\ c+D/2,\ 1-\sigma-D/2;\ 1-b+\sigma, 1-a+\sigma$	$\frac{r^2}{p^2};\ \frac{q^2}{p^2};\ \frac{m^2}{q^2}$
8	$a+D/2,\ -b,\ 1-\sigma-D/2;\ 1-b+\sigma, 1+a-\sigma$	$\frac{r^2}{p^2};\ \frac{q^2}{p^2};\ \frac{m^2}{q^2}$
9	$-c,\ b+D/2,\ b-\sigma;\ b+D/2,\ 1+b-c$	$-\frac{p^2q^2}{m^2r^2};\ \frac{p^2}{r^2};\ -\frac{m^2}{p^2}$
10	$-b,\ c+D/2,\ c-\sigma;\ c+D/2,\ 1-b+c$	$-\frac{p^2q^2}{m^2r^2};\ \frac{q^2}{r^2};\ -\frac{m^2}{q^2}$
11	$-c,\ b+D/2,\ 1-c-D/2;\ 1-c+\sigma,\ 1+b-c$	$-\frac{m^2}{r^2};\ \frac{q^2}{r^2};\ -\frac{p^2}{m^2}$
12	$-b,\ c+D/2,\ 1-b-D/2;\ 1-b+\sigma,\ 1-b+c$	$-\frac{m^2}{r^2};\ \frac{p^2}{r^2};\ -\frac{q^2}{m^2}$
13	$c+D/2,\ b+D/2,\ 1-b-D/2;\ 1-b+\sigma,\ 1-a+\sigma$	$\frac{r^2}{p^2};\ \frac{m^2}{p^2};\ \frac{q^2}{m^2}$
14	$-b,\ -c,\ 1-b-D/2;\ 1-b+\sigma,\ 1+a-\sigma$	$\frac{r^2}{p^2};\ \frac{m^2}{p^2};\ \frac{q^2}{m^2}$
15	$b+D/2,\ c+D/2,\ 1-c-D/2;\ 1-c+\sigma,\ 1-a+\sigma$	$\frac{r^2}{q^2};\ \frac{m^2}{q^2};\ \frac{p^2}{m^2}$
16	$-c,\ -b,\ 1-c-D/2;\ 1-c+\sigma,\ 1+a-\sigma$	$\frac{r^2}{q^2};\ \frac{m^2}{q^2};\ \frac{p^2}{m^2}$
17	$-c,\ b+D/2,\ 1-\sigma-D/2;\ 1-c+\sigma,\ 1+b-\sigma$	$\frac{q^2}{r^2};\ \frac{p^2}{r^2};\ -\frac{m^2}{p^2}$
18	$-b,\ c+D/2,\ 1-\sigma-D/2;\ 1-b+\sigma,\ 1+c-\sigma$	$\frac{p^2}{r^2};\ \frac{q^2}{r^2};\ -\frac{m^2}{q^2}$
19	$-a,\ b+D/2,\ 1-\sigma-D/2;\ 1+b-\sigma,\ 1-a+\sigma$	$\frac{q^2}{p^2};\ \frac{r^2}{p^2};\ \frac{m^2}{p^2}$
20	$-\sigma,\ -c,\ 1-\sigma-D/2;\ 1+b-\sigma,\ 1+a-\sigma$	$\frac{q^2}{p^2};\ \frac{r^2}{p^2};\ \frac{m^2}{p^2}$
21	$-a,\ c+D/2,\ 1-\sigma-D/2;\ 1+c-\sigma,\ 1-a+\sigma$	$\frac{p^2}{q^2};\ \frac{r^2}{q^2};\ \frac{m^2}{q^2}$
22	$-\sigma,\ -b,\ 1-\sigma-D/2;\ 1+c-\sigma,\ 1+a-\sigma$	$\frac{p^2}{q^2};\ \frac{r^2}{q^2};\ \frac{m^2}{q^2}$
23	$-a,\ -\sigma,\ 1-\sigma-D/2;\ 1+b-\sigma,\ 1+c-\sigma$	$\frac{p^2}{r^2};\ \frac{q^2}{r^2};\ -\frac{m^2}{r^2}$

Table 9.

n	$1,\cdots,6$	$7,8$	$9,\cdots,12$	$13,\cdots,16$	$17,18$	$19,\cdots,22$	$23,\cdots,26$	$27,28$	29
l	1	2	3	4	5	6	7	8	9

$$= \sum_{j_1,..,j_4=0}^{\infty} \frac{(x_1)_{j_1+j_2+j_3}(x_2)_{j_1+j_2-j_4}(x_3)_{j_2+j_3}}{(x_4)_{j_2-j_4}(x_5)_{j_1+j_2+j_3-j_4}} \times \frac{z_1^{j_1}}{j_1!}\frac{z_2^{j_2}}{j_2!}\frac{z_3^{j_3}}{j_3!}\frac{z_4^{j_4}}{j_4!},$$

$$R_2\left[\begin{array}{l} x_1,x_2,x_3 \\ x_4,x_5 \end{array}\middle| z_1;z_2;z_3;z_4\right] = \sum_{j_1,..,j_4=0}^{\infty} \frac{(x_1)_{j_1+j_2-j_3-j_4}(x_2)_{j_1+j_2}(x_3)_{j_3+j_4}}{(x_4)_{j_1-j_3}(x_5)_{j_2-j_4}} \times \frac{z_1^{j_1}}{j_1!}\frac{z_2^{j_2}}{j_2!}\frac{z_3^{j_3}}{j_3!}\frac{z_4^{j_4}}{j_4!},$$

$$R_3\left[\begin{array}{l} x_1,x_2,x_3 \\ x_4,x_5 \end{array}\middle| z_1;z_2;z_3;z_4\right] = \sum_{j_1,..,j_4=0}^{\infty} \frac{(x_1)_{j_1+j_2-j_3-j_4}(x_2)_{j_1+j_2-j_3}(x_3)_{j_3+j_4}}{(x_4)_{j_2-j_3}(x_5)_{j_1-j_3-j_4}} \times \frac{z_1^{j_1}}{j_1!}\frac{z_2^{j_2}}{j_2!}\frac{z_3^{j_3}}{j_3!}\frac{z_4^{j_4}}{j_4!},$$

$$R_4\left[\begin{array}{l} x_1,x_2,x_3 \\ x_4,x_5 \end{array}\middle| z_1;z_2;z_3;z_4\right] = \sum_{j_1,..,j_4=0}^{\infty} \frac{(x_1)_{j_1+j_2-j_3}(x_2)_{j_1+j_4}(x_3)_{j_2-j_4}}{(x_4)_{j_2-j_3-j_4}(x_5)_{j_1-j_3}}$$

$$\times \frac{z_1^{j_1}}{j_1!}\frac{z_2^{j_2}}{j_2!}\frac{z_3^{j_3}}{j_3!}\frac{z_4^{j_4}}{j_4!},$$

$$R_5\left[\begin{matrix} x_1, x_2, x_3 \\ x_4, x_5 \end{matrix}\middle| z_1; z_2; z_3; z_4\right]$$

$$= \sum_{j_1,..,j_4=0}^{\infty} \frac{(x_1)_{j_1+j_2+j_3}(x_2)_{j_1+j_3}(x_3)_{j_2+j_4}}{(x_4)_{j_1+j_2+j_4}(x_5)_{j_3-j_4}} \frac{z_1^{j_1}}{j_1!}\frac{z_2^{j_2}}{j_2!}\frac{z_3^{j_3}}{j_3!}\frac{z_4^{j_4}}{j_4!},$$

$$R_6\left[\begin{matrix} x_1, x_2, x_3 \\ x_4, x_5 \end{matrix}\middle| z_1; z_2; z_3; z_4\right]$$

$$= \sum_{j_1,..,j_4=0}^{\infty} \frac{(x_1)_{j_1+j_2+j_3+j_4}(x_2)_{j_1+j_2+j_3}(x_3)_{j_1+j_4}}{(x_4)_{j_1+j_3+j_4}(x_5)_{j_1+j_2}}$$

$$\times \frac{z_1^{j_1}}{j_1!}\frac{z_2^{j_2}}{j_2!}\frac{z_3^{j_3}}{j_3!}\frac{z_4^{j_4}}{j_4!},$$

$$R_7\left[\begin{matrix} x_1, x_2, x_3 \\ x_4, x_5 \end{matrix}\middle| z_1; z_2; z_3; z_4\right]$$

$$= \sum_{j_1,..,j_4=0}^{\infty} \frac{(x_1)_{j_1+j_2+j_3}(x_2)_{j_1+j_2-j_4}(x_3)_{j_3+j_4}}{(x_4)_{j_2-j_4}(x_5)_{j_1+j_3}}$$

$$\times \frac{z_1^{j_1}}{j_1!}\frac{z_2^{j_2}}{j_2!}\frac{z_3^{j_3}}{j_3!}\frac{z_4^{j_4}}{j_4!},$$

$$R_8\left[\begin{matrix} x_1, x_2, x_3 \\ x_4, x_5 \end{matrix}\middle| z_1; z_2; z_3; z_4\right]$$

$$= \sum_{j_1,..,j_4=0}^{\infty} \frac{(x_1)_{j_1+j_2-j_3}(x_2)_{j_1+j_2}(x_3)_{j_3+j_4}}{(x_4)_{j_1-j_3-j_4}(x_5)_{j_2+j_4}} \frac{z_1^{j_1}}{j_1!}\frac{z_2^{j_2}}{j_2!}\frac{z_3^{j_3}}{j_3!}\frac{z_4^{j_4}}{j_4!},$$

$$R_9\left[\begin{matrix} x_1, x_2, x_3 \\ x_4, x_5 \end{matrix}\middle| z_1; z_2; z_3; z_4\right]$$

$$= \sum_{j_1,..,j_4=0}^{\infty} \frac{(x_1)_{j_1+j_2+j_3+j_4}(x_2)_{j_1+j_2}(x_3)_{j_3+j_4}}{(x_4)_{j_1+j_3}(x_5)_{j_2+j_4}}$$

$$\times \frac{z_1^{j_1}}{j_1!}\frac{z_2^{j_2}}{j_2!}\frac{z_3^{j_3}}{j_3!}\frac{z_4^{j_4}}{j_4!},$$

$$T_9\left[\begin{matrix} x_1, x_2, x_3, x_4, x_5 \\ x_6, x_7 \end{matrix}\middle| z_1, z_2; z_3\right]$$

$$= \sum_{j_1,j_2,j_3=0}^{\infty} \frac{(x_1)_{j_1+j_2}(x_2)_{j_2+j_3}(x_3)_{-j_1+j_3}(x_4)_{-2j_1+j_3}(x_5)_{j_1}}{(x_6)_{-j_1+j_2+j_3}(x_7)_{-j_1+j_3}}$$

$$\times \frac{z_1^{j_1}}{j_1!}\frac{z_2^{j_2}}{j_2!}\frac{z_3^{j_3}}{j_3!},$$

$$T_{10}\left[\begin{matrix} x_1, x_2, x_3, x_4, x_5 \\ x_6, x_7 \end{matrix}\middle| z_1, z_2; z_3\right]$$

$$= \sum_{j_1,j_2,j_3=0}^{\infty} \frac{(x_1)_{j_1+j_2+j_3}(x_2)_{j_1+j_2}(x_3)_{j_1+j_3}(x_4)_{j_2+j_3}(x_5)_{j_3}}{(x_6)_{j_1+j_2+j_3}(x_7)_{j_1+j_2+2j_3}}$$

$$\times \frac{z_1^{j_1}}{j_1!}\frac{z_2^{j_2}}{j_2!}\frac{z_3^{j_3}}{j_3!}$$

Table 10.

n	C_n
1	$\pi^{D/2}(p^2)^\sigma \frac{(-a)_\sigma(-b)_\sigma}{(-\sigma)_{2\sigma+D/2}}$
2	$\pi^{D/2}(q^2)^\sigma \frac{(-a)_\sigma(-c)_\sigma}{(-\sigma)_{2\sigma+D/2}}$
3	$\pi^{D/2}(r^2)^\sigma \frac{(-b)_\sigma(-c)_\sigma}{(-\sigma)_{2\sigma+D/2}}$
4	$\pi^{D/2}(p^2)^\sigma(z_2)^{-c} \frac{(-a)_{-c+\sigma}(-b)_\sigma}{(c-\sigma)_{-c+2\sigma+D/2}}$
5	$\pi^{D/2}(q^2)^\sigma(z_2)^{-b} \frac{(-a)_{-b+\sigma}(-c)_\sigma}{(b-\sigma)_{-b+2\sigma+D/2}}$
6	$\pi^{D/2}(p^2)^\sigma(z_3)^{-c} \frac{(-a)_\sigma(-b)_{-c+\sigma}}{(c-\sigma)_{-c+2\sigma+D/2}}$
7	$\pi^{D/2}(q^2)^\sigma(z_3)^{-b} \frac{(-a)_\sigma(-c)_{-b+\sigma}}{(b-\sigma)_{-b+2\sigma+D/2}}$
8	$\pi^{D/2}(r^2)^\sigma(z_1)^{-a} \frac{(-b)_{-a+\sigma}(-c)_\sigma}{(a-\sigma)_{-a+2\sigma+D/2}}$
9	$\pi^{D/2}(r^2)^\sigma(z_1)^{-a} \frac{(-b)_\sigma(-c)_{-a+\sigma}}{(a-\sigma)_{-a+2\sigma+D/2}}$
10	$\pi^{D/2}(p^2)^\sigma(z_1)^{\sigma-a}(z_2)^{\sigma-b} \frac{(-a)_{-c-D/2}(-b)_{-c-D/2}}{(c+D/2)_{-2c-D/2}}$
11	$\pi^{D/2}(q^2)^\sigma(z_1)^{\sigma-a}(z_2)^{\sigma-c} \frac{(-a)_{-b-D/2}(-c)_{-b-D/2}}{(b+D/2)_{-2b-D/2}}$
12	$\pi^{D/2}(r^2)^\sigma(z_1)^{\sigma-b}(z_2)^{\sigma-c} \frac{(-b)_{-a-D/2}(-c)_{-a-D/2}}{(a+D/2)_{-2a-D/2}}$
13	$\pi^{D/2}(p^2)^\sigma(z_1)^{\sigma-b}(-z_2)^{\sigma-a} \frac{(-a)_{-c-D/2}(-b)_{-c-D/2}}{(c+D/2)_{-2c-D/2}}$
14	$\pi^{D/2}(q^2)^\sigma(z_1)^{\sigma-c}(-z_2)^{\sigma-a} \frac{(-a)_{-b-D/2}(-c)_{-b-D/2}}{(b+D/2)_{-2b-D/2}}$
15	$\pi^{D/2}(r^2)^\sigma(z_1)^{\sigma-c}(-z_4)^{\sigma-b} \frac{(-b)_{-a-D/2}(-c)_{-a-D/2}}{(a+D/2)_{-2a-D/2}}$
16	$\pi^{D/2}(r^2)^\sigma(z_1)^{\sigma-b}(-z_4)^{\sigma-c} \frac{(-b)_{-a-D/2}(-c)_{-a-D/2}}{(a+D/2)_{-2a-D/2}}$
17	$\pi^{D/2}(-m^2)^\sigma(z_3)^{-c}(z_4)^{-a} \frac{(-a)_c}{(a-\sigma)_{c+D/2}}$
18	$\pi^{D/2}(-m^2)^\sigma(z_3)^{-b}(z_4)^{-a} \frac{(-a)_b}{(a-\sigma))_{b+D/2}}$
19	$\pi^{D/2}(-m^2)^\sigma(-z_1)^{-b}(z_4)^{-a} \frac{(-b)_a}{(b-\sigma)_{a+D/2}}$
20	$\pi^{D/2}(-m^2)^\sigma(-z_1)^{-c}(z_4)^{-a} \frac{(-c)_a}{(c-\sigma)_{a+D/2}}$
21	$\pi^{D/2}(-m^2)^\sigma(-z_3)^{-c}(-z_4)^{-a} \frac{1}{(-b-D/2)_{D/2}}$
22	$\pi^{D/2}(-m^2)^\sigma(-z_3)^{-b}(-z_4)^{-a} \frac{1}{(-c-D/2)_{D/2}}$
23	$\pi^{D/2}(-m^2)^\sigma(z_4)^{c-\sigma} \frac{(-a)_{2a+D/2}(-b)_{2b+D/2}}{(c-\sigma)_{-2c+2\sigma+D/2}}$
24	$\pi^{D/2}(-m^2)^\sigma(z_4)^{b-\sigma} \frac{(-a)_{2a+D/2}(-c)_{2c+D/2}}{(b-\sigma)_{-2b+2\sigma+D/2}}$
25	$\pi^{D/2}(-m^2)^\sigma(z_2)^{-b} \frac{(-a)_b(-c)_{-b+\sigma}}{(b-\sigma)_{\sigma+D/2}}$
26	$\pi^{D/2}(-m^2)^\sigma(z_2)^{-c} \frac{(-a)_c(-b)_{-c+\sigma}}{(c-\sigma)_{\sigma+D/2}}$
27	$\pi^{D/2}(-m^2)^\sigma(-z_1)^{-c} \frac{(-b)_{-a-D/2}}{(-b+\sigma)_{-a}}$
28	$\pi^{D/2}(-m^2)^\sigma(-z_1)^{-b} \frac{(-c)_{-a-D/2}}{(-c+\sigma)_{-a}}$
29	$\pi^{D/2}(-m^2)^\sigma(-z_1)^{a+D/2} \frac{(-b)_{-a-D/2}(-c)_{-a-D/2}}{(a+D/2)_{-2a-D/2}}$
30	$\pi^{D/2}(-m^2)^\sigma(-z_3)^{-a} \frac{(-b)_a}{(a-\sigma)_{a+D/2}}$
31	$\pi^{D/2}(-m^2)^\sigma(-z_3)^{-a} \frac{(-c)_a}{(a-\sigma)_{a+D/2}}$
32	$\pi^{D/2}(-m^2)^\sigma \frac{(D/2)_a}{(-\sigma)_{a+D/2}}$

Table 11.

n	$x_1,\, x_2,\, x_3;\, x_4,\, x_5$	$z_1; z_2; z_3; z_4$
1	$-\sigma,\, -c,\, 1-\sigma-D/2;\, 1+a-\sigma,\, 1+b-\sigma$	$-\frac{m^2}{p^2};\ \frac{q^2}{p^2};\ \frac{r^2}{p^2};\ \frac{m^2}{p^2}$
2	$-\sigma,\, -b,\, 1-\sigma-D/2;\, 1+a-\sigma,\, 1+c-\sigma$	$-\frac{m^2}{q^2};\ \frac{p^2}{q^2};\ \frac{r^2}{q^2};\ \frac{m^2}{q^2}$
3	$-\sigma,\, -a,\, 1-\sigma-D/2;\, 1+c-\sigma,\, 1+b-\sigma$	$\frac{p^2}{r^2};\ \frac{q^2}{r^2};\ \frac{m^2}{r^2};\ \frac{m^2}{r^2}$
4	$-c,\, b+D/2,\, 1-\sigma-D/2;\, 1-c+\sigma,\, 1+b-\sigma$	$\frac{q^2}{r^2};\ \frac{p^2}{r^2};\ \frac{m^2}{r^2};\ \frac{m^2}{p^2}$
5	$-b,\, c+D/2,\, 1-\sigma-D/2;\, 1-b+\sigma,\, 1+c-\sigma$	$\frac{p^2}{r^2};\ \frac{q^2}{r^2};\ \frac{m^2}{r^2};\ \frac{m^2}{q^2}$
6	$-c,\, a+D/2,\, 1-\sigma-D/2;\, 1+a-\sigma,\, 1-c+\sigma$	$\frac{r^2}{q^2};\ \frac{m^2}{q^2};\ \frac{p^2}{q^2};\ -\frac{m^2}{p^2}$
7	$-b,\, a+D/2,\, 1-\sigma-D/2;\, 1+a-\sigma,\, 1-b+\sigma$	$\frac{r^2}{p^2};\ \frac{m^2}{p^2};\ \frac{q^2}{p^2};\ -\frac{m^2}{q^2}$
8	$c+D/2,\, -a,\, 1-\sigma-D/2;\, 1-a+\sigma,\, 1+c-\sigma$	$\frac{r^2}{q^2};\ \frac{p^2}{q^2};\ \frac{m^2}{r^2};\ -\frac{m^2}{r^2}$
9	$b+D/2,\, -a,\, 1-\sigma-D/2;\, 1-a+\sigma,\, 1+b-\sigma$	$\frac{r^2}{p^2};\ \frac{q^2}{p^2};\ \frac{m^2}{r^2};\ -\frac{m^2}{r^2}$
10	$\sigma+D/2,\, c+D/2,\, 1-\sigma-D/2;\, 1-b+\sigma,\, 1-a+\sigma$	$\frac{r^2}{p^2};\ \frac{q^2}{p^2};\ -\frac{m^2p^2}{q^2r^2};\ \frac{m^2}{r^2}$
11	$\sigma+D/2,\, b+D/2,\, 1-\sigma-D/2;\, 1-c+\sigma,\, 1-a+\sigma$	$\frac{r^2}{q^2};\ \frac{p^2}{q^2};\ -\frac{m^2q^2}{p^2r^2};\ \frac{m^2}{r^2}$
12	$\sigma+D/2,\, a+D/2,\, 1-\sigma-D/2;\, 1-b+\sigma,\, 1-c+\sigma$	$\frac{q^2}{r^2};\ \frac{p^2}{r^2};\ \frac{m^2}{q^2};\ \frac{m^2}{p^2}$
13	$c+D/2,\, a+D/2,\, 1-a-D/2;\, 1-a+\sigma,\, 1-b+\sigma$	$\frac{q^2}{p^2};\ \frac{m^2}{p^2};\ -\frac{p^2}{q^2};\ \frac{r^2}{m^2}$
14	$b+D/2,\, a+D/2,\, 1-a-D/2;\, 1-a+\sigma,\, 1-c+\sigma$	$\frac{p^2}{q^2};\ \frac{m^2}{q^2};\ -\frac{q^2}{p^2};\ \frac{r^2}{m^2}$
15	$b+D/2,\, a+D/2,\, b-\sigma;\, b+D/2,\, 1-c+\sigma$	$\frac{p^2}{r^2};\ \frac{q^2}{m^2};\ -\frac{m^2}{p^2};\ \frac{m^2}{r^2}$
16	$c+D/2,\, a+D/2,\, c-\sigma;\, c+D/2, 1-b+\sigma$	$\frac{q^2}{r^2};\ \frac{p^2}{m^2};\ -\frac{m^2}{q^2};\ \frac{m^2}{r^2}$
17	$-c,\, a-\sigma,\, a+D/2;\, a+D/2,\, 1+a-c$	$\frac{p^2}{q^2};\ -\frac{p^2r^2}{m^2q^2};\ \frac{p^2}{q^2};\ -\frac{m^2}{p^2}$
18	$-b,\, a-\sigma,\, a+D/2;\, a+D/2,\, 1+a-b$	$\frac{q^2}{p^2};\ -\frac{q^2r^2}{m^2p^2};\ \frac{q^2}{p^2};\ -\frac{m^2}{q^2}$
19	$a-b,\, 1-b-D/2,\, -a;\, 1-b+\sigma,\, 1-b-D/2$	$\frac{m^2}{r^2};\ \frac{m^2}{r^2};\ -\frac{q^2r^2}{m^2p^2};\ \frac{r^2}{p^2}$
20	$a-c,\, 1-c-D/2,\, -a;\, 1-c+\sigma,\, 1-c-D/2$	$\frac{m^2}{r^2};\ \frac{m^2}{r^2};\ -\frac{p^2r^2}{m^2q^2};\ \frac{r^2}{q^2}$
21	$1+b-\sigma,\, -c,\, -a;\, 1+b+D/2,\, 1+b-\sigma$	$\frac{-m^2q^2}{p^2r^2};\ \frac{m^2}{r^2};\ \frac{m^2}{r^2};\ \frac{m^2}{p^2}$
22	$1+c-\sigma,\, -a,\, -b;\, 1+c+D/2,\, 1+c-\sigma$	$-\frac{m^2p^2}{q^2r^2};\ \frac{m^2}{r^2};\ \frac{m^2}{r^2};\ \frac{m^2}{q^2}$
23	$-c,\, b+D/2,\, a+D/2;\, 1-c+\sigma,\, a+b+D$	$\frac{q^2}{m^2};\ -\frac{p^2}{m^2};\ \frac{r^2}{m^2};\ -\frac{m^2}{p^2}$
24	$-b,\, c+D/2,\, a+D/2;\, 1-b+\sigma,\, a+c+D$	$\frac{p^2}{m^2};\ -\frac{q^2}{m^2};\ \frac{r^2}{m^2};\ -\frac{m^2}{q^2}$
25	$-b,\, 1-b-D/2,\, a+D/2;\, 1-b+\sigma,\, 1+a-b$	$\frac{m^2}{p^2};\ -\frac{m^2}{p^2};\ \frac{r^2}{p^2};\ -\frac{q^2}{m^2}$
26	$-c,\, 1-c-D/2,\, a+D/2;\, 1-c+\sigma,\, 1+a-c$	$\frac{m^2}{q^2};\ -\frac{m^2}{q^2};\ \frac{r^2}{q^2};\ -\frac{p^2}{m^2}$
27	$-c,\, 1-c-D/2,\, -a;\, 1-c+\sigma,\, 1+b-\sigma$	$\frac{m^2}{r^2};\ \frac{m^2}{r^2};\ \frac{q^2}{r^2};\ \frac{p^2}{m^2}$
28	$-b,\, 1-b-D/2,\, -a;\, 1-b+\sigma,\, 1+c-\sigma$	$\frac{m^2}{r^2};\ \frac{m^2}{r^2};\ \frac{p^2}{r^2};\ \frac{q^2}{m^2}$
29	$1+a,\, a+D/2,\, -a;\, 1-c+\sigma,\, 1-b+\sigma$	$\frac{m^2}{r^2};\ \frac{m^2}{r^2};\ \frac{p^2}{m^2};\ \frac{q^2}{m^2}$

n	$x_1,\, x_2,\, x_3,\, x_4,\, x_5;\, x_6,\, x_7$	$z_1,\, z_2,\, z_3$
30	$-c,\, -a,\, 1-a-D/2,\, 1-a+b+c,\, a+D/2;\, 1-a+b,\, 1-a+\sigma$	$-\frac{r^2}{m^2};\ \frac{q^2}{p^2};\ \frac{m^2}{p^2}$
31	$-b,\, -a,\, 1-a-D/2,\, 1-a+b+c,\, a+D/2;\, 1-a+c,\, 1-a+\sigma$	$-\frac{r^2}{m^2};\ \frac{p^2}{q^2};\ \frac{m^2}{q^2}$
32	$-\sigma,\, -a,\, -b,\, -c,\, a+D/2;\, D/2,\, -b-c$	$\frac{p^2}{m^2};\ \frac{q^2}{m^2};\ \frac{r^2}{m^2}$

Table 12.

n	$1,\cdots,15$	$16,\cdots,21$	$22,\cdots,27$	$28,\cdots,33$
l	1	2	3	4

Table 13.

n	D_n
1	$\pi^{D/2}(r^2)^\sigma(-z_4)^{\sigma-c}(-z_5)^{\sigma-b}\frac{(-b)_{-a-D/2}(-c)_{-a-D/2}}{(a+D/2)_{-2a-D/2}}$
2	$\pi^{D/2}(q^2)^\sigma(-z_4)^{\sigma-a}(-z_5)^{\sigma-c}\frac{(-a)_{-b-D/2}(-c)_{-b-D/2}}{(b+D/2)_{-2b-D/2}}$
3	$\pi^{D/2}(p^2)^\sigma(-z_4)^{\sigma-b}(-z_5)^{\sigma-a}\frac{(-a)_{-c-D/2}(-b)_{-c-D/2}}{(c+D/2)_{-2c-D/2}}$
4	$\pi^{D/2}(-m^2)^\sigma(z_3)^{-c}(z_4)^{-a}\frac{(-b)_{-c-D/2}}{(-a+c)_{-c}}$
5	$\pi^{D/2}(-m^2)^\sigma(z_3)^{-b}(z_4)^{-a}\frac{(-c)_{-b-D/2}}{(-a+b)_{-b}}$
6	$\pi^{D/2}(-m^2)^\sigma(z_3)^{-c}(z_4)^{-b}\frac{(-a)_{-c-D/2}}{(-b+c)_{-c}}$
7	$\pi^{D/2}(-m^2)^\sigma(z_3)^{-a}(z_4)^{-b}\frac{(-c)_{-a-D/2}}{(-b+a)_{-a}}$
8	$\pi^{D/2}(-m^2)^\sigma(z_3)^{-b}(z_4)^{-c}\frac{(-a)_{-b-D/2}}{(-c+b)_{-b}}$
9	$\pi^{D/2}(-m^2)^\sigma(z_3)^{-a}(z_4)^{-c}\frac{(-b)_{-a-D/2}}{(-c+a)_{-a}}$
10	$\pi^{D/2}(-m^2)^\sigma(z_5)^{c+D/2}\frac{(-a)_{-c-D/2}(-b)_{-c-D/2}}{(c+D/2)_{-2c-D/2}}$
11	$\pi^{D/2}(-m^2)^\sigma(z_5)^{c-\sigma}\frac{(-a)_{2a+D/2}(-b)_{2b+D/2}}{(c-\sigma)_{-2c+2\sigma+D/2}}$
12	$\pi^{D/2}(-m^2)^\sigma(z_5)^{b+D/2}\frac{(-a)_{-b-D/2}(-c)_{-b-D/2}}{(b+D/2)_{-2b-D/2}}$
13	$\pi^{D/2}(-m^2)^\sigma(z_5)^{b-\sigma}\frac{(-a)_{2a+D/2}(-c)_{2c+D/2}}{(b-\sigma)_{-2b+2\sigma+D/2}}$
14	$\pi^{D/2}(-m^2)^\sigma(z_5)^{a-\sigma}\frac{(-b)_{2b+D/2}(-c)_{2c+D/2}}{(a-\sigma)_{-2a+2\sigma+D/2}}$
15	$\pi^{D/2}(-m^2)^\sigma(z_5)^{a+D/2}\frac{(-b)_{-a-D/2}(-c)_{-a-D/2}}{(a+D/2)_{-2a-D/2}}$
16	$\pi^{D/2}(p^2)^\sigma(z_4)^{-c}\frac{(-a)_\sigma(-b)_{-c+\sigma}}{(c-\sigma)_{-c+2\sigma+D/2}}$
17	$\pi^{D/2}(q^2)^\sigma(z_4)^{-b}\frac{(-a)_\sigma(-c)_{-b+\sigma}}{(b-\sigma)_{-b+2\sigma+D/2}}$
18	$\pi^{D/2}(p^2)^\sigma(z_4)^{-c}\frac{(-a)_{-c+\sigma}(-b)_\sigma}{(c-\sigma)_{-c+2\sigma+D/2}}$
19	$\pi^{D/2}(r^2)^\sigma(z_4)^{-a}\frac{(-b)_\sigma(-c)_{-a+\sigma}}{(a-\sigma)_{-a+2\sigma+D/2}}$
20	$\pi^{D/2}(q^2)^\sigma(z_4)^{-b}\frac{(-a)_{-b+\sigma}(-c)_\sigma}{(b-\sigma)_{-b+2\sigma+D/2}}$
21	$\pi^{D/2}(r^2)^\sigma(z_4)^{-a}\frac{(-b)_{-a+\sigma}(-c)_\sigma}{(a-\sigma)_{-a+2\sigma+D/2}}$
22	$\pi^{D/2}(-m^2)^\sigma(-z_4)^{-b}(-z_5)^{-c}\frac{1}{(-a-D/2)_{D/2}}$
23	$\pi^{D/2}(-m^2)^\sigma(-z_4)^{-a}(-z_5)^{-c}\frac{1}{(-b-D/2)_{D/2}}$
24	$\pi^{D/2}(-m^2)^\sigma(-z_4)^{-a}(-z_5)^{-b}\frac{1}{(-c-D/2)_{D/2}}$
25	$\pi^{D/2}(p^2)^\sigma\frac{(-a)_\sigma(-b)_\sigma}{(-\sigma)_{2\sigma+D/2}}$
26	$\pi^{D/2}(q^2)^\sigma\frac{(-a)_\sigma(-c)_\sigma}{(-\sigma)_{2\sigma+D/2}}$
27	$\pi^{D/2}(r^2)^\sigma\frac{(-b)_\sigma(-c)_\sigma}{(-\sigma)_{2\sigma+D/2}}$

Table 13. (continued)

n	D_n
28	$\pi^{D/2}(-m^2)^\sigma(z_1)^{c+D/2}(-z_3)^{-a}\frac{(-a)_{-c-D/2}(-b)_{-c-D/2}}{(c+D/2)_{-2c-D/2}}$
29	$\pi^{D/2}(-m^2)^\sigma(z_1)^{b+D/2}(-z_3)^{-a}\frac{(-a)_{-b-D/2}(-c)_{-b-D/2}}{(b+D/2)_{-2b-D/2}}$
30	$\pi^{D/2}(-m^2)^\sigma(z_4)^{c+D/2}(z_5)^{-b}\frac{(-a)_{-c-D/2}(-b)_{-c-D/2}}{(c+D/2)_{-2c-D/2}}$
31	$\pi^{D/2}(-m^2)^\sigma(z_4)^{a+D/2}(z_5)^{-b}\frac{(-b)_{-a-D/2}(-c)_{-a-D/2}}{(a+D/2)_{-2a-D/2}}$
32	$\pi^{D/2}(-m^2)^\sigma(z_4)^{b+D/2}(z_5)^{-c}\frac{(-a)_{-b-D/2}(-c)_{-b-D/2}}{(b+D/2)_{-2b-D/2}}$
33	$\pi^{D/2}(-m^2)^\sigma(z_4)^{a+D/2}(z_5)^{-c}\frac{(-b)_{-a-D/2}(-c)_{-a-D/2}}{(a+D/2)_{-2a-D/2}}$
34	$\pi^{D/2}(-m^2)^\sigma(-z_1)^{-b}\frac{(-a)_{-c-D/2}}{(-a+b)_{-c}}$
35	$\pi^{D/2}(-m^2)^\sigma(-z_1)^{-a}\frac{(-b)_{-c-D/2}}{(a-b)_{-c}}$
36	$\pi^{D/2}(-m^2)^\sigma(-z_1)^{-c}\frac{(-a)_{-b-D/2}}{(-a+c)_{-b}}$
37	$\pi^{D/2}(-m^2)^\sigma(-z_1)^{-a}\frac{(-c)_{-b-D/2}}{(a-c)_{-b}}$
38	$\pi^{D/2}(-m^2)^\sigma(-z_1)^{-c}\frac{(-b)_{-a-D/2}}{(-b+c)_{-a}}$
39	$\pi^{D/2}(-m^2)^\sigma(-z_1)^{-b}\frac{(-c)_{-a-D/2}}{(b-c)_{-a}}$
40	$\pi^{D/2}(-m^2)^\sigma\frac{1}{(-\sigma)_{D/2}}$

and the expression for each one solution is given by $J_n = C_n R_l$, $J_{30} = C_{30}T_9$, $J_{31} = C_{31}T_9$, $J_{32} = C_{32}T_{10}$ where the relation between n and l is given by Table 9, where the coefficients C_n are shown in Table 10 and the parameters and variables of the R_l and T_l functions in Table 11.

D Three massive denominators

The three massive denominators solutions $J_n = J_n(a,b,c,D,p,q,m,m,m)$, where $n = 1,2,\cdots,40$, are given by hypergeometric functions listed below:

$$\Psi_1\left[\begin{array}{c} x_1,x_2,x_3 \\ x_4,x_5 \end{array}\middle| z_1;z_2;z_3;z_4;z_5\right] = \sum_{j_1,..,j_5=0}^{\infty}\frac{(x_1)_{j_1+j_2+j_3}(x_2)_{j_1+j_2-j_4}(x_3)_{j_1+j_3-j_5}}{(x_4)_{j_1-j_4-j_5}(x_5)_{j_1+j_2+j_3-j_4-j_5}} \times\frac{z_1^{j_1}}{j_1!}\frac{z_2^{j_2}}{j_2!}\frac{z_3^{j_3}}{j_3!}\frac{z_4^{j_4}}{j_4!}\frac{z_5^{j_5}}{j_5!},$$

$$\Psi_2\left[\begin{array}{c} x_1,x_2,x_3 \\ x_4,x_5 \end{array}\middle| z_1;z_2;z_3;z_4;z_5\right] = \sum_{j_1,..,j_5=0}^{\infty}\frac{(x_1)_{j_1+j_2+j_5}(x_2)_{j_3+j_4+j_5}(x_3)_{j_1+j_2-j_4}}{(x_4)_{j_2-j_3-j_4}(x_5)_{j_1+j_3+j_5}} \times\frac{z_1^{j_1}}{j_1!}\frac{z_2^{j_2}}{j_2!}\frac{z_3^{j_3}}{j_3!}\frac{z_4^{j_4}}{j_4!}\frac{z_5^{j_5}}{j_5!},$$

Table 14.

n	x_1, x_2, x_3; x_4, x_5	$z_1; z_2; z_3; z_4; z_5$
1	$1-\sigma-D/2,\ c-\sigma,\ b-\sigma;\ 1-a-D/2,\ 1-\sigma-D/2$	$-\frac{m^2r^2}{p^2q^2};\ \frac{m^2}{p^2};\ \frac{m^2}{q^2};\ -\frac{p^2}{r^2};\ -\frac{q^2}{r^2}$
2	$1-\sigma-D/2,\ a-\sigma,\ c-\sigma;\ 1-b-D/2,\ 1-\sigma-D/2$	$-\frac{m^2q^2}{p^2r^2};\ \frac{m^2}{r^2};\ \frac{m^2}{p^2};\ -\frac{r^2}{q^2};\ -\frac{p^2}{q^2}$
3	$1-\sigma-D/2,\ b-\sigma,\ a-\sigma;\ 1-c-D/2,\ 1-\sigma-D/2$	$-\frac{m^2p^2}{q^2r^2};\ \frac{m^2}{q^2};\ \frac{m^2}{r^2};\ -\frac{q^2}{p^2};\ -\frac{r^2}{p^2}$
4	$-c,\ a+D/2,\ a-\sigma;\ a+D/2,\ 1+a-c$	$-\frac{p^2r^2}{m^2q^2};\ \frac{p^2}{q^2};\ \frac{p^2}{q^2};\ -\frac{m^2}{p^2};\ -\frac{m^2}{p^2}$
5	$-b,\ a+D/2,\ a-\sigma;\ a+D/2,\ 1+a-b$	$-\frac{q^2r^2}{m^2p^2};\ \frac{q^2}{p^2};\ \frac{q^2}{p^2};\ -\frac{m^2}{q^2};\ -\frac{m^2}{q^2}$
6	$-c,\ b-\sigma,\ b+D/2;\ b+D/2,\ 1-c+b$	$-\frac{p^2q^2}{m^2r^2};\ \frac{p^2}{r^2};\ \frac{p^2}{r^2};\ -\frac{m^2}{p^2};\ -\frac{m^2}{p^2}$
7	$-a,\ b-\sigma,\ b+D/2;\ b+D/2,\ 1-a+b$	$-\frac{q^2r^2}{m^2p^2};\ \frac{r^2}{p^2};\ \frac{r^2}{p^2};\ -\frac{m^2}{r^2};\ -\frac{m^2}{r^2}$
8	$-b,\ c-\sigma,\ c+D/2;\ c+D/2,\ 1-b+c$	$-\frac{p^2q^2}{m^2r^2};\ \frac{q^2}{r^2};\ \frac{q^2}{r^2};\ -\frac{m^2}{q^2};\ -\frac{m^2}{q^2}$
9	$-a,\ c-\sigma,\ c+D/2;\ c+D/2,\ 1-a+c$	$-\frac{p^2r^2}{m^2q^2};\ \frac{r^2}{q^2};\ \frac{r^2}{q^2};\ -\frac{m^2}{r^2};\ -\frac{m^2}{r^2}$
10	$-c,\ b-\sigma,\ a-\sigma;\ 1-c-D/2,\ -c$	$-\frac{p^2}{m^2};\ \frac{q^2}{m^2};\ \frac{r^2}{m^2};\ -\frac{m^2}{p^2};\ -\frac{m^2}{p^2}$
11	$-c,\ b+D/2,\ a+D/2,\ 1-c+\sigma,\ a+b+D$	$-\frac{p^2}{m^2};\ \frac{q^2}{m^2};\ \frac{r^2}{m^2};\ -\frac{m^2}{p^2};\ -\frac{m^2}{p^2}$
12	$-b,\ c-\sigma,\ a-\sigma;\ 1-b-D/2,\ -b$	$-\frac{q^2}{m^2};\ \frac{p^2}{m^2};\ \frac{r^2}{m^2};\ -\frac{m^2}{q^2};\ -\frac{m^2}{q^2}$
13	$-b,\ c+D/2,\ a+D/2;\ 1-b+\sigma,\ a+c+D$	$-\frac{q^2}{m^2};\ \frac{p^2}{m^2};\ \frac{r^2}{m^2};\ -\frac{m^2}{q^2};\ -\frac{m^2}{q^2}$
14	$-a,\ c+D/2,\ b+D/2;\ 1-a+\sigma,\ b+c+D$	$-\frac{r^2}{m^2};\ \frac{p^2}{m^2};\ \frac{q^2}{m^2};\ -\frac{m^2}{r^2};\ -\frac{m^2}{r^2}$
15	$-a,\ c-\sigma,\ b-\sigma;\ 1-a-D/2,\ -a$	$-\frac{r^2}{m^2};\ \frac{p^2}{m^2};\ \frac{q^2}{m^2};\ -\frac{m^2}{r^2};\ -\frac{m^2}{r^2}$
16	$1-\sigma-D/2,\ a+D/2,\ -c;\ 1-c+\sigma,\ 1+a-\sigma$	$\frac{m^2}{p^2};\ \frac{m^2}{p^2};\ -\frac{r^2}{q^2};\ \frac{p^2}{q^2};\ \frac{m^2}{q^2}$
17	$1-\sigma-D/2,\ -b,\ b-\sigma;\ 1-a-D/2, 1+a-\sigma$	$\frac{m^2}{q^2};\ \frac{m^2}{q^2};\ -\frac{r^2}{p^2};\ \frac{q^2}{p^2};\ \frac{m^2}{p^2}$
18	$1-\sigma-D/2,\ -c,\ c-\sigma;\ 1-b-D/2,\ 1+b-\sigma$	$\frac{m^2}{p^2};\ \frac{m^2}{p^2};\ -\frac{q^2}{r^2};\ \frac{p^2}{r^2};\ \frac{m^2}{r^2}$
19	$1-\sigma-D/2,\ -a,\ a-\sigma;\ 1-b-D/2,\ 1+b-\sigma$	$\frac{m^2}{r^2};\ \frac{m^2}{r^2};\ -\frac{q^2}{p^2};\ \frac{r^2}{p^2};\ \frac{m^2}{p^2}$
20	$1-\sigma-D/2,\ -b,\ b-\sigma;\ 1-c-D/2,\ 1+c-\sigma$	$\frac{m^2}{q^2};\ \frac{m^2}{q^2};\ -\frac{p^2}{r^2};\ \frac{q^2}{r^2};\ \frac{m^2}{r^2}$
21	$1-\sigma-D/2, -a, a-\sigma, 1-c-D/2, 1-\sigma+c$	$\frac{m^2}{r^2};\ \frac{m^2}{r^2};\ -\frac{p^2}{q^2};\ \frac{r^2}{q^2};\ \frac{m^2}{q^2}$
22	$1+a-\sigma,\ -b,\ -c;\ 1+a+D/2,\ 1+a-\sigma$	$-\frac{m^2r^2}{p^2q^2};\ \frac{m^2}{p^2};\ \frac{m^2}{q^2};\ \frac{m^2}{p^2};\ \frac{m^2}{q^2}$
23	$1+b-\sigma,\ -a,\ -c;\ 1+b+D/2,\ 1+b-\sigma$	$-\frac{m^2q^2}{p^2r^2};\ \frac{m^2}{p^2};\ \frac{m^2}{r^2};\ \frac{m^2}{p^2};\ \frac{m^2}{r^2}$
24	$1+c-\sigma,\ -a,\ -b;\ 1+c+D/2,\ 1+c-\sigma$	$-\frac{m^2p^2}{q^2r^2};\ \frac{m^2}{q^2};\ \frac{m^2}{r^2};\ \frac{m^2}{q^2};\ \frac{m^2}{r^2}$
25	$-\sigma,\ -c, 1-\sigma-D/2;\ 1+a-\sigma,\ 1+b-\sigma$	$-\frac{m^2}{p^2};\ \frac{q^2}{p^2};\ \frac{m^2}{p^2};\ \frac{r^2}{p^2};\ \frac{m^2}{p^2}$
26	$-\sigma,\ -b,\ 1-\sigma-D/2;\ 1+a-\sigma,\ 1+c-\sigma$	$-\frac{m^2}{q^2};\ \frac{p^2}{q^2};\ \frac{m^2}{q^2};\ \frac{r^2}{q^2};\ \frac{m^2}{q^2}$
27	$-\sigma,\ -a,\ 1-\sigma-D/2;\ 1+b-\sigma,\ 1+c-\sigma$	$-\frac{m^2}{r^2};\ \frac{p^2}{r^2};\ \frac{m^2}{r^2};\ \frac{q^2}{r^2};\ \frac{m^2}{r^2}$
28	$a-\sigma, a+D/2,\ c+D/2;\ a+D/2;\ 1-b+\sigma$	$\frac{r^2}{m^2};\ \frac{p^2}{q^2};\ \frac{m^2}{p^2};\ \frac{q^2}{p^2};\ -\frac{m^2}{q^2}$
29	$a+D/2, a-\sigma, c-\sigma;\ a+D/2, 1-b-D/2$	$\frac{r^2}{m^2};\ \frac{p^2}{q^2};\ \frac{m^2}{p^2};\ \frac{q^2}{p^2};\ -\frac{m^2}{q^2}$
30	$b+D/2, b-\sigma, a-\sigma;,\ b+D/2, 1-c-D/2$	$\frac{q^2}{m^2};\ \frac{r^2}{p^2};\ \frac{m^2}{r^2};\ \frac{p^2}{r^2};\ -\frac{m^2}{p^2}$
31	$b-\sigma,\ b+D/2,\ a+D/2;\ b+D/2,\ 1-c+\sigma$	$\frac{q^2}{m^2};\ \frac{r^2}{p^2};\ \frac{m^2}{r^2};\ \frac{p^2}{r^2};\ -\frac{m^2}{p^2}$
32	$c-\sigma,\ c+D/2,\ b+D/2;\ c+D/2,\ 1-a+\sigma$	$\frac{p^2}{m^2};\ \frac{q^2}{r^2};\ \frac{m^2}{q^2};\ \frac{r^2}{q^2};\ -\frac{m^2}{r^2}$
33	$c+D/2, c-\sigma, b-\sigma;\ c+D/2, 1-a-D/2$	$\frac{p^2}{m^2};\ \frac{q^2}{r^2};\ \frac{m^2}{q^2};\ \frac{r^2}{q^2};\ -\frac{m^2}{r^2}$

Table 14. (continued)

n	$x_1,\, x_2,\, x_3,\, x_4;\, x_5,\, x_6,\, x_7$	$z_1;\, z_2;\, z_3;\, z_4$
34	$-b,\, 1-b-D/2,\, 1+a-b+c,\, -c;\, 1+a-b,\, 1-b-D/2,\, 1-b+\sigma$	$\frac{m^2}{p^2};\, \frac{m^2}{p^2};\, \frac{r^2}{p^2};\, \frac{q^2}{m^2}$
35	$-a,\, 1-a-D/2,\, 1-a+b+c,\, -c;\, 1-a+b,\, 1-a-D/2,\, 1-a+\sigma$	$\frac{m^2}{p^2};\, \frac{m^2}{p^2};\, \frac{q^2}{p^2};\, \frac{r^2}{m^2}$
36	$-c,\, 1-c-D/2,\, 1+a+b-c,\, -b;\, 1+a-c,\, 1-c-D/2,\, 1-c+\sigma$	$\frac{m^2}{q^2};\, \frac{m^2}{q^2};\, \frac{r^2}{q^2};\, \frac{p^2}{m^2}$
37	$-a,\, 1-a-D/2,\, 1-a+b+c,\, -b;\, 1-a+c,\, 1-a-D/2,\, 1-a+\sigma$	$\frac{m^2}{q^2};\, \frac{m^2}{q^2};\, \frac{p^2}{q^2};\, \frac{r^2}{m^2}$
38	$-c,\, 1-c-D/2,\, 1+a+b-c,\, -a;\, 1+b-c,\, 1-c-D/2,\, 1-c+\sigma$	$\frac{m^2}{r^2};\, \frac{m^2}{r^2};\, \frac{q^2}{r^2};\, \frac{p^2}{m^2}$
39	$-b,\, 1-b-D/2,\, 1+a-b+c,\, -a;\, 1-b+c,\, 1-b-D/2,\, 1-b+\sigma$	$\frac{m^2}{r^2};\, \frac{m^2}{r^2};\, \frac{p^2}{r^2};\, \frac{q^2}{m^2}$
n	$x_1,\, x_2,\, x_3,\, x_4;\, x_5$	$z_1;\, z_2;\, z_3$
40	$-\sigma,\, -a,\, -b,\, -c;\, -\sigma+D/2$	$\frac{p^2}{m^2};\, \frac{q^2}{m^2};\, \frac{r^2}{m^2}$

$$\Psi_3\left[\begin{matrix} x_1,x_2,x_3 \\ x_4,x_5 \end{matrix}\middle| z_1;z_2;z_3;z_4;z_5\right] = \sum_{j_1,\ldots,j_5=0}^{\infty} \frac{(x_1)_{j_1+j_2+j_3+j_4+j_5}(x_2)_{j_1+j_2+j_4}(x_3)_{j_1+j_3+j_5}}{(x_4)_{j_1+j_4+j_5}(x_5)_{j_1+j_2+j_3}} \times \frac{z_1^{j_1}}{j_1!}\frac{z_2^{j_2}}{j_2!}\frac{z_3^{j_3}}{j_3!}\frac{z_4^{j_4}}{j_4!}\frac{z_5^{j_5}}{j_5!},$$

$$\Psi_4\left[\begin{matrix} x_1,x_2,x_3 \\ x_4,x_5 \end{matrix}\middle| z_1;z_2;z_3;z_4;z_5\right] = \sum_{j_1,\ldots,j_5=0}^{\infty} \frac{(x_1)_{j_1+j_2-j_3}(x_2)_{j_1+j_4-j_5}(x_3)_{-j_2+j_3+j_4}}{(x_4)_{j_1-j_3-j_5}(x_5)_{-j_2+j_4-j_5}} \times \frac{z_1^{j_1}}{j_1!}\frac{z_2^{j_2}}{j_2!}\frac{z_3^{j_3}}{j_3!}\frac{z_4^{j_4}}{j_4!}\frac{z_5^{j_5}}{j_5!},$$

$$R_{10}\left[\begin{matrix} x_1,x_2,x_3,x_4 \\ x_5,x_6,x_7 \end{matrix}\middle| z_1;z_2;z_3;z_4\right] = \sum_{j_1,\ldots,j_4=0}^{\infty} \frac{(x_1)_{j_1+j_2+j_3}(x_2)_{j_1+j_2-j_4}(x_3)_{j_1+j_2-2j_4}(x_4)_{j_3+j_4}}{(x_5)_{j_1+j_2+j_3-j_4}(x_6)_{j_2-j_4}(x_7)_{j_1-j_4}} \times \frac{z_1^{j_1}}{j_1!}\frac{z_2^{j_2}}{j_2!}\frac{z_3^{j_3}}{j_3!}\frac{z_4^{j_4}}{j_4!},$$

$$T_{11}\left[\begin{matrix} x_1,x_2,x_3,x_4 \\ x_5 \end{matrix}\middle| z_1;z_2;z_3\right] = \sum_{j_1,j_2,j_3=0}^{\infty} \frac{(x_1)_{j_1+j_2+j_3}(x_2)_{j_1+j_2}(x_3)_{j_1+j_3}}{(x_5)_{2j_1+2j_2+2j_3}} \times (x_4)_{j_2+j_3}\frac{z_1^{j_1}}{j_1!}\frac{z_2^{j_2}}{j_2!}\frac{z_3^{j_3}}{j_3!},$$

and the expression of each one solution is given by $J_n = D_n\Psi_l$, where the relation between n and l is given by Table 12, or $J_n = D_nR_{10}$, $n = 34,\cdots,39$; $J_{40} = D_{40}T_{11}$, where the coefficients D_n are shown in Table 13, the parameters and variables of the functions Ψ_l, R_{10}, and T_{11} in Table 14.

References

1. G. 't Hooft, M. Veltman, Nucl. Phys. B **44**, 189 (1972)
2. C.G. Bollini, J.J. Giambiagi, Nuovo Cimento B, **12**, 20 (1972)
3. É.É. Boos, A.I. Davydychev, Theor. Math. Phys. **89**, 1052 (1991)
4. A.E. Terrano, Phys. Lett. B **93**, 424 (1980)
5. F.V. Tkachov, Phys. Lett. B **100**, 65 (1981)
6. V.A. Smirnov, Nucl. Phys. B **566**, 469 (2000); Phys. Lett. B **460**, 397 (1999)
7. S. Laporta, E. Remiddi, Phys. Lett. B **356**, 390 (1995); Phys. Lett. B **379**, 283 (1996); V.W. Hughes, T. Kinoshita, Rev. Mod. Phys. **71**, S133 (1999); S. Laporta, E. Remiddi, Acta Phys. Pol. B **28**, 959 (1997)
8. C. Anastasiou, E.W.N. Glover, C. Oleari, Nucl. Phys. B **575**, 416 (2000); Erratum ibid. B **585**, 763 (2000)
9. Z. Bern, L. Dixon, D.A. Kosower, JHEP **1**, 27 (2000)
10. T. Gehrmann, E. Remiddi, Nucl. Phys. B **601**, 287 (2001); Nucl. Phys. Proc. Suppl. **89**, 251 (2000)
11. K. Chetyrkin, M. Misiak, M. Münz, Nucl. Phys. B **518**, 473 (1998)
12. E.W.N. Glover, J.B. Tausk, J.J. van der Bij, Phys. Lett. B **516**, 33 (2001)
13. J.B. Tausk, Phys. Lett. B **469**, 225 (1999)
14. J. Fleischer, V.A. Smirnov, A. Frink, J. Körner, D. Kreimer, K. Schilcher, J.B. Tausk, Eur. Phys. J. C **2**, 747 (1998)
15. I.G. Halliday, R.M. Ricotta, Phys. Lett. B **193**, 241 (1987); G.V. Dunne, I.G. Hallyday, Phys. Lett. B **193**, 248 (1987)
16. A.T. Suzuki, A.G.M. Schmidt, J. Phys. A Math. Gen. **31**, 8023 (1998)
17. A.T. Suzuki, A.G.M. Schmidt, Eur. Phys. J. C **12**, 361 (2000)
18. A.T. Suzuki, A.G.M. Schmidt, Eur. Phys. J. C **5**, 175 (1998)
19. A.T. Suzuki, A.G.M. Schmidt, Phys. Lett. B **494**, 332 (2000)

20. A.I. Davydychev, M.Yu. Kalmykov, hep-th/0012189; A.I. Davydychev, A.G. Grozin, Eur. Phys. J. C **020**, 333 (2001) (hep-ph/0103078); A.I. Davydychev, P. Osland, L. Saks, JHEP **0108**, 050 (2001) (hep-ph/0105072)
21. D. Melikhov, S. Simula, hep-ph/0112044
22. G. 't Hooft, M. Veltman, Nucl. Phys. B **153**, 365 (1979)
23. G. Passarino, M. Veltman, Nucl. Phys. B **160**, 151 (1979)
24. F.T. Brandt, J. Frenkel, Phys. Rev. D **33**, 464 (1986)
25. L.G. Cabral-Rosetti, M.A. Sanchis-Lozano, hep-ph/0206081
26. G.J. Van Oldenborgh, J.A.M. Vermaseren, Z. Phys. C **46**, 425 (1990)
27. Y.L. Luke, The special functions and their approximations, Vol. I (Academic Press, 1969); L.J. Slater, Generalized hypergeometric functions (Cambridge University Press, 1966); P. Appel, J. Kampé de Feriet, Fonctions hypergéométriques et hypersphériques. Polynomes d'Hermite (Gauthier-Villars, Paris 1926); A. Erdélyi, W. Magnus, F. Oberhettinger, F.G. Tricomi, Higher transcendental functions (McGraw-Hill, 1953)